Engineering Notes

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Fitting Atmospheric Parameters Using Parabolic Blending

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Introduction

A COMPUTATIONALLY fast and accurate fit of atmospheric parameters is needed when designing computer simulations for propulsive devices, examining flight characteristics of various shapes, or determining the trajectory of an orbiting body during atmospheric entry. With the possibility of interplanetary exploration, these simulations need not be limited to the Earth's atmosphere. A round trip to Mars, for example, involves a great deal of descent/ascent modeling based on limited atmospheric data. The two most important properties that must be determined are the relationships of pressure and density to altitude, with other properties such as temperature and the speed of sound derived from them. Problems involving drag or re-entry trajectories will invariably require atmospheric density, just as thrust-related problems will require atmospheric pressure. Currently, Chebyshev polynomials[1] are used in many numerical simulations to yield approximations of these parameters. The computer graphics technique of parabolic blending[2] is explored as an alternative method of fitting atmospheric data.

Present Method

The currently accepted method of modeling an atmosphere for pressure and density characteristics uses a Chebyshev polynomial expansion,[1] where the desired parameter of an altitude is calculated from the product of the standard mean sea level pressure or density and a corresponding ratio. This ratio r is determined by polynomial coefficients a_k, through the following equation:

$$r = e^{\frac{1}{2}[a_0 + \sum_{k=1}^{n} a_k C_k(\eta)]} \tag{1}$$

with the running variables $C_k(\eta)$ established by a recursive relationship, where η is a function of altitude. Two different sets of polynomial coefficients a_k are used to represent atmospheric pressure and density. These coefficients are precomputed to make the Chebyshev polynomial accurately fit a specified atmosphere for a certain altitude range, thereby limiting the expansion's use to that particular atmosphere and range.

The Chebyshev expansion for the 1962 atmosphere from sea level to 80 km, used in this paper as a basis for comparison, is accurate within 0.25% for pressure and within 2% for density. Although such accuracy is impressive, this method has several shortcomings. When modeling an atmosphere with limited data, such as that of Mars, extrapolation may be necessary. A Chebyshev expansion is only valid for interpolation because the polynomial fit is based on a weighted sum of sinusoids, with the coefficients determining the appropriate weights to closely match the empirical data. Beyond the endpoints of the model, these weights cause the curve to depart rapidly, rendering the expansion of little use. Also, the Chebyshev expansion is accurate only for the atmosphere for which the coefficients were derived. In the case of this particular expansion, the ratio coefficients were derived from the empirical data for the 1962 standard atmosphere, so that the polynomial can only yield pressures and densities for that atmosphere. Additionally, this method is not easily modified to accommodate updates to empirical data, again limiting the use of the expansion to a particular atmosphere. It is possible to update the Chebyshev coefficients, but to do so requires the use of a statistical technique, such as a least-squares differential correction or some variant thereof. This process can be a complicated and tedious one.

The Chebyshev expansion must compute a new pressure or density ratio for each altitude of interest. This process requires 29 products, 29 sums, and one logarithmic operation, excluding the one-time computational overhead to find the polynomial coefficients. The time required to complete this operation is not significant when finding parameters at a single altitude. However, if the computation must be completed many times, as in analyzing the ascent trajectory of a rocket, the time until completion becomes a concern.

Proposed Method

The need for an improved computational model of an atmosphere is driven by the desire for timeliness and accuracy. To alleviate the problems associated with the Chebyshev fit, a computer graphics technique known as parabolic blending is employed. This method, developed by A. W. Overhauser[2] and refined by Brewer and Anderson,[3] is reformulated to establish a direct relationship between intermediate abscissa and ordinate values while accounting for unequally spaced points.

The atmospheric characteristics around any locus of altitudes can be approximated using a series of polynomial functions, with three adjacent points defining a parabola. This method of parabolic blending uses four consecutive points to create two second-order curve functions, one defining the first three points and another defining the latter three points, both having two points in common. These two functions are then blended into a single third-order polynomial that represents the curve between the second and third points. By careful selection of the blending equation, the cubic polynomial will match the slope of the first parabola at the second point and the slope of the second parabola at the third point, as shown in Fig. 1. The result is a smooth curve between points two and three, with no discontinuity in slope at those points. This method is repeated for all points in the set, resulting in one continuous curve consisting of numerous localized cubic polynomials. It is this localization that enables the user to easily change the model by changing the points in the empirical data set. Also, the oscillations often induced by high-order polyno-

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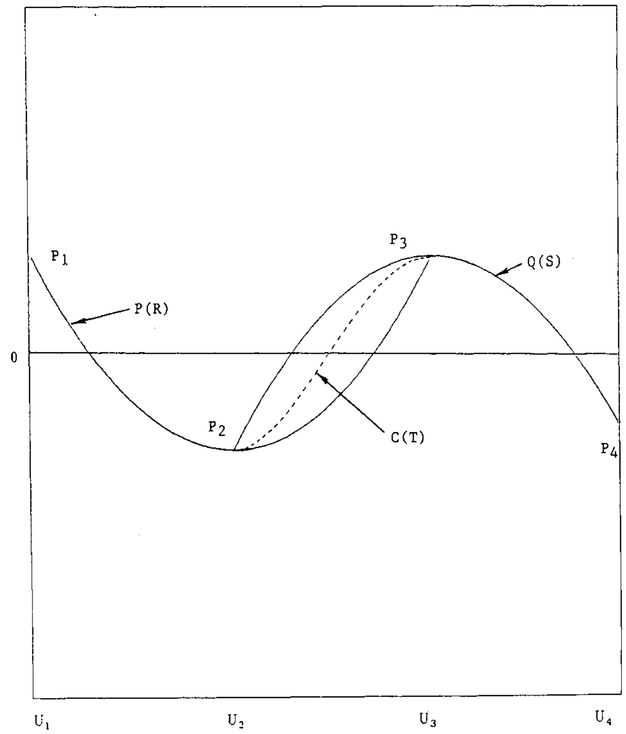

Fig. 1 Notation for parabolic blending.

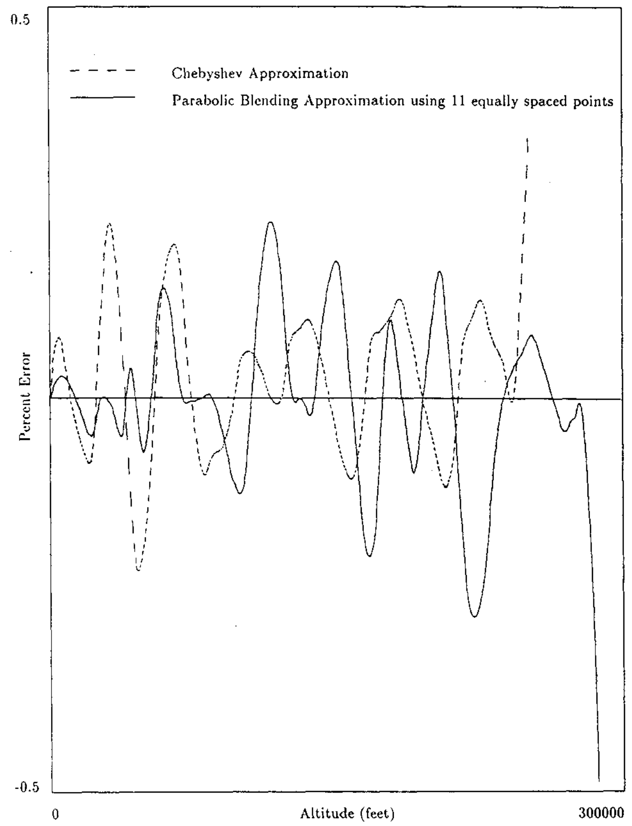

Fig. 2 Pressure error vs altitude (parabolic blending approximation using 11 equally spaced points).

mial fits are greatly reduced because the localization limits the blending function to third order.

Blending is accomplished by using four consecutive point pairs of altitude and corresponding pressure or density, (U_1, P_1),...,(U_4, P_4), as seen in Fig. 1. The first parabola P is a function of the unitized parameter R:

$$P(R) = A_P R^2 + B_P R + C_P \tag{2}$$

where R ranges from 0 to 1 as the abscissa of interest ranges from U_1 to U_3. The polynomial coefficients are computed using the following relationships:

$$\tau = \frac{U_2 - U_1}{U_3 - U_1} \tag{3a}$$

$$A_P = \frac{P_2}{\tau(\tau - 1)} + \frac{-P_3}{(\tau - 1)} + \frac{P_1}{(\tau)} \tag{3b}$$

$$B_P = P_3 - P_1 - A_P \tag{3c}$$

$$C_P = P_1 \tag{3d}$$

In Eqs. (3), τ corresponds to the value of R at U_2. In a similar manner, the second parabola Q is a function of the unitized parameter S:

$$Q(S) = A_Q S^2 + B_Q S + C_Q \tag{4}$$

where S ranges from zero to one as the abscissa of interest ranges from U_2 to U_4.

The C curve, ranging from U_2 to U_3, weights the P and Q curves through the blending function:

$$U_2 \le U \le U_3$$

$$T = \frac{U - U_2}{U_3 - U_2}$$

$$C(T) = (1 - T)P(R) + TQ(S) \tag{5}$$

To determine C from U, four consecutive points must be picked from the model's data file, with two of the points having abscissa values less than U and two having values greater than U. Determine the P and Q curve coefficients, then the values of R, S, and T. With these values known, compute $P(R)$, $Q(S)$, and, finally, $C(T)$. It is interesting to note that while operating between the second and third points, the P and Q curve coefficients remain constant and need be computed only once. Should four consecutive points meeting the stated criteria not exist, such as near the data file's endpoints, then blending is not accomplished and only one parabolic curve is computed for interpolation. For limited extrapolation, the single parabolic curve is also used, allowing R to exceed its lower bound or S to exceed its upper bound.

Computationally, this process requires up to 19 products and 28 sums. If the P and Q coefficients from a previous iteration are still valid, then only 9 products and 12 sums are needed. Also, if near an endpoint, the computations require 8 products and 12 sums. Since the decrease of atmospheric pressure and density are exponentially related to altitude, accuracy can be improved by taking the natural logarithm of the ordinal points, as is done in the Chebyshev model. The blending method must then take the antilog to arrive at the proper value, adding one logarithmic operation to the computations.

Comparison of Results

To compare the performance of parabolic blending to the well accepted Chebyshev expansion, 15 empirical data points from the 1962 table were used in the blending file, equaling the number of Chebyshev coefficients. Eleven equally spaced altitudes, ranging from 0 to 250,000 ft, were entered in the blending file, leaving four altitudes to be chosen to zero out maximum error points wherever they occurred, thus demonstrating the power the user has to tailor the blending file and improve accuracy.

Figure 2 shows the pressure errors of the Chebyshev model (dashed line) and the 11 point parabolic blending method (solid line), with Fig. 3 showing the density errors. Both have similar accuracies within the altitude bounds of 250,000 ft,

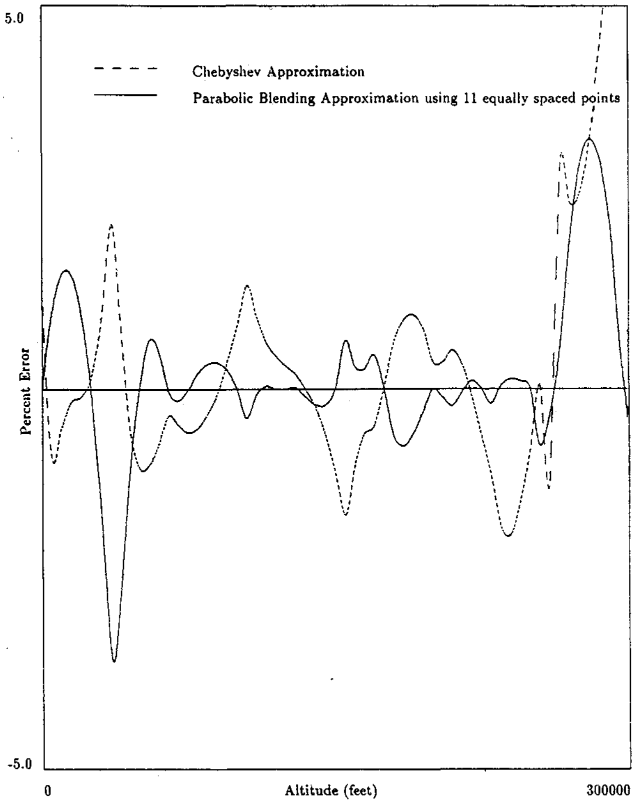

Fig. 3 Density error vs altitude (parbolic blending approximation using 11 equally spaced points).

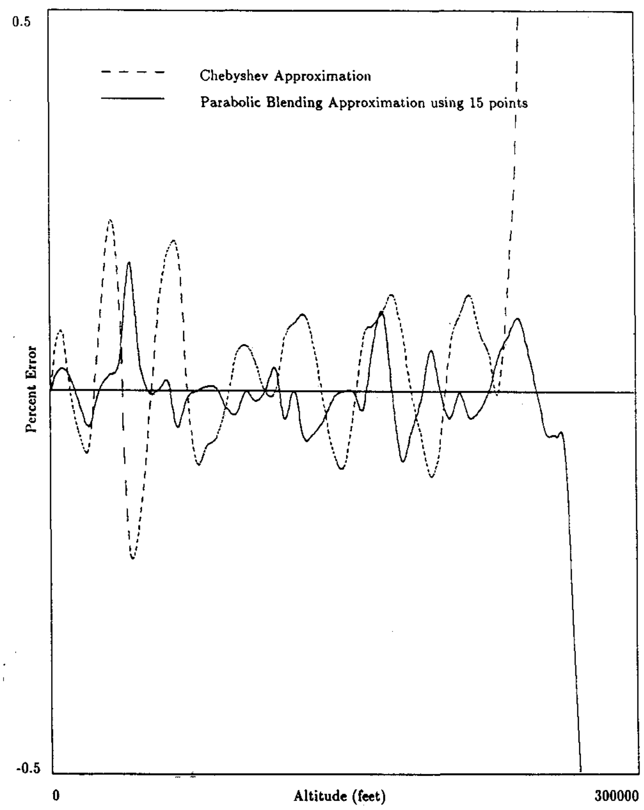

Fig. 4 Pressure error vs altitude (parabolic blending approximation using 15 equally spaced points).

with the blended curve error increasing more slowly when out of bounds, thereby lending itself to limited extrapolation. Because the blending scheme is interpolative and allows local curve control, it is a simple task to zero out any error by

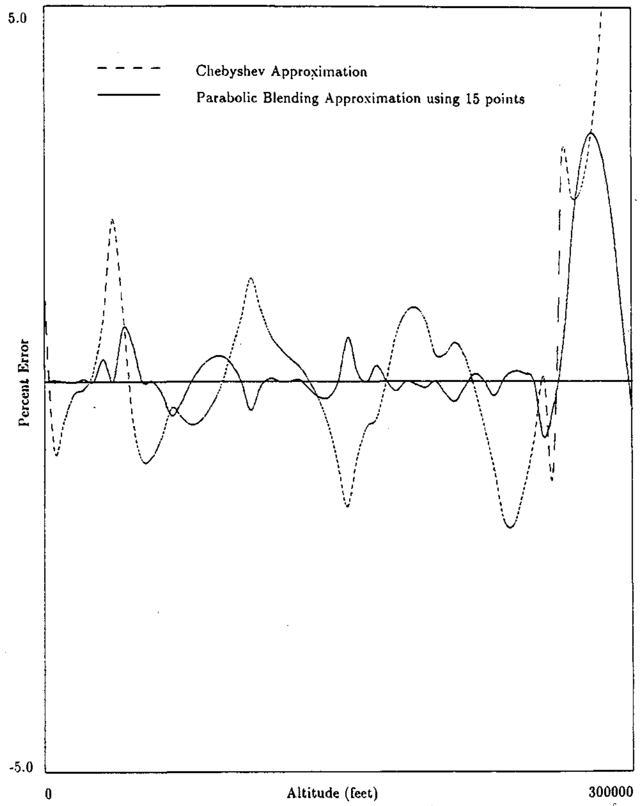

Fig. 5 Density error vs altitude (parabolic blending approximation using 15 equally spaced points).

adding the corresponding point pair to the blending file. The results of such tailoring are seen in Figs. 4 and 5 when the remaining four points are added. By including the entire atmospheric data set, if available, all inbound errors are completely eliminated.

The parabolic blending fit is almost twice as fast as the Chebyshev algorithm. As more points are added to the blending file, the search time for the required consecutive points increases, but the computational burden associated with blending remains the same. The user has the option of trading speed for fitting accuracy by simply adding atmospheric data points to the blending file.

Conclusions

When compared to the Chebyshev polynomial expansion, the parabolic blending method can be user tailored to work over any range of altitudes, producing faster results while accurately fitting the atmospheric table values for pressure and density. Accuracy can be further improved by adding additional data points to the blending file, with some extrapolation of results outside the altitude limits also being possible. The model can be easily updated as more empirical data become available, or quickly changed if an entirely different atmosphere is desired. In contrast, the Chebyshev model is limited to the range of altitudes for which its coefficients were designed, and modifications can only be made through a complicated and time-consuming process. In summary, the parabolic blending fit is fast and flexible without sacrificing accuracy.

References

[1]"U.S. Standard Atmosphere Supplements, 1966," U.S. Government Printing Office, Washington, D.C., 1966, pp. 67–78, 171, 173.

[2]Overhauser, A. W., "Analytic Definition of Curves and Surfaces by Parabolic Blending," Ford Motor Co. Scientific Lab., Dearborn, MI, TR No. SL68-40, May 8, 1968.

[3]Brewer, J. A., and Anderson, D. C., "Visual Interaction with Overhauser Curves and Surfaces," *Computer Graphics*, SIGGRAPH-ACM, Vol. 11, No. 2, Summer 1977, pp. 132-137.